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TECHNICAL MEMORANDUM

THE MICROSTRUCTURE OF TURBULENT FLOW*

By A. M. Obukhoff and A. M. Yaglom

In 1941 a general theory of locally isotropic turbulence was proposed by Kolmogoroff which permitted the prediction of a number of laws of turbulent flow for large Reynolds numbers. The most important of these laws, the dependence of the mean square of the difference in velocities at two points on their distance and the dependence of the coefficient of turbulence diffusion on the scale of the phenomenon, were obtained by both Kolmogoroff (references 1 and 2) and Obukhoff (reference 3) in the same year. At the present time these laws have been experimentally confirmed by direct measurements carried out in aerodynamic wind tunnels in the laboratory (references 4 and 5), in the atmosphere (references 6 and 7), and also on the ocean (reference 8).[1] In recent years in the Laboratory of Atmospheric Turbulence of the Geophysics Institute of the Soviet Academy of Sciences, a number of investigations have been conducted in which this theory was further developed. The results of several of these investigations are presented in this paper.[2]

The fundamental physical concepts which are the basis of Kolmogoroff's theory may briefly by summarized as follows.[3] A turbulent flow at large Reynolds numbers is considered to be the result of the imposing of disturbances (vortices or eddies) of all possible scales of

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*"Mikrostructura turbulentnogo potoka," Prikladnaya Matematika i Mekhanika, Vol. XV, 1951, pp. 3-26.

[1]The applications of these laws to certain problems of the physics of the atmosphere may be found in references 9 and 10.

[2]In addition to the results contained in the present article, reference may also be made to the theoretical investigation of the structure of the temperature field (or of the concentrations of any neutral additive) in the turbulent flow, presented in references 11 and 12. The applications of the latter results may be found in references 13 and 14.

[3]For a more detailed presentation see reference 15.

magnitude. Only the very largest of these vortices arise directly from the instability of the mean flow. The scale L of these large vortices is comparable with the distance over which the velocity of the mean flow changes (for example, in a turbulent boundary layer, with the distance from the wall).[4]

The motion of the largest vortices is unstable and gives rise to smaller vortices of the second order; vortices of the second order give rise to still smaller vortices of the third order, and so forth, down to the smallest vortices which are stable (i.e. the characterizing Reynolds number is less than the critical value). Since for all vortices, except the smallest ones, the characteristic Reynolds number is large, the viscosity has no appreciable effect on their motion. The motion of all vortices that are not too small is therefore not associated with any marked dissipation of energy; the vortices of the n^{th} order use practically all the energy which is received from the vortices of the $(n-1)^{th}$ order to form the vortices of the $(n+1)^{th}$ order. However, the motion of the smallest of the existing vortices is "laminar" and depends essentially on the molecular viscosity. In these very small vortices the entire energy that is transferred along the vortex cascade goes over into heat energy.

The motion of all the vortices, except for the very largest, may be assumed homogeneous and isotropic. Any directional effect of the mean flow ceases to be appreciable for vortices of a relatively low order. It is also of importance that this motion may be assumed quasi-stationary, that is, a change in the statistical characteristics of the motion of the vortices under consideration proceeds very slowly in comparison with the periods characteristic of these vortices. It follows that the motion of all vortices whose scales are considerably less than L (the microstructure or local structure of the flow) must be subject to certain general statistical laws which do not depend on the geometry of the flow and on the properties of the mean flow. The establishment of these general laws, which have a wide range of applicability, constitutes the theory of local isotropic turbulence.

In the investigation of the laws of the local structure, considerations from the theories of similitude and dimensions are of great value. It is only these considerations which permit obtaining a number of essential results. To apply these ideas it is necessary, first of all, to separate out those fundamental magnitudes on which the local structure of the flow may depend. On account of the homogeneous and isotropic character of the motion of the vortex system under consideration, the

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[4]The length L coincides with the length of the mixing path introduced in the semiempirical theory of turbulence.

characteristics of the mean motion (of the type of length characteristics, velocity characteristics, etc.) do not enter among these fundamental magnitudes. Therefore, only two magnitudes remain, the mean dissipation of energy per unit time per unit mass of the fluid ε, which determines the intensity of the energy flow transferred along a cascade of vortices of different scales, and the kinematic viscosity ν, which plays an essential role in the process of dissipation.[5] These two magnitudes thus play a fundamental part in the theory that is presented herein.

The dimensions of ε and ν are:

$$[\varepsilon] = L^2T^{-3}$$

$$[\nu] = L^2T^{-1}$$

From these two magnitudes, it is evidently possible to form a single combination in the dimension of length

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

The length η determines an internal scale characteristic of the local structure. By use of the previously described physical picture of turbulent motion, it is possible to identify η with the scale of the smallest vortices in which a dissipation of energy occurs (since this picture does not contain any other characteristic length). The scale η was first introduced in the work of Kolmogoroff (reference 1); it is termed the internal (or local) scale of turbulence (in contrast to the external scale L).

In the further analysis of the microstructure, two limiting cases may be considered separately to advantage: the case of scales much larger than η and that of scales much smaller than η. First, the system of vortices with dimensions much smaller than L but much greater than the scale η of the smallest vortices is considered. The motion of these vortices, as has already been pointed out, should not depend

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[5] The fluid is assumed everywhere to be incompressible and to have a constant density ρ. The magnitude ρ is not included herein among the fundamental magnitudes because in the main part of the paper (sections 1 and 3), the purely kinematic characteristics of the flow, which of course cannot depend on the density, will be considered. When, however, the structure of the pressure field (section 2) is investigated, it is necessary to add ρ to ε and ν. Information on the fundamental magnitudes on which the local structure of the temperature field may depend is found in references 11 and 12.

on the viscosity ν, a circumstance which immediately facilitates the obtaining of concrete results by computation of the dimensions. In the second extreme case, for scales of motion much less than η, the motion may be assumed laminar. However, in the intermediate range of scales of the order of η, the theory of dimensions gives, as a rule, less concrete results. Thus, for example, it follows from this theory that any nondimensional function of the distance determined by the local structure should be a universal function of r/η. The form of this function for values of the argument of the order of unity remains however undetermined.

In the present paper an attempt is made to describe quantitatively the structure of the fundamental hydrodynamic fields (pressure, velocity, and acceleration[6]) for all distances less than L (i.e., for the entire range for which the theory of Kolmogoroff applies). For this purpose some additional hypotheses are introduced which have a certain experimental basis. The asymptotic formulas for $r>>\eta$ and for $r<<\eta$ obtained are in agreement with known earlier results where all the undetermined numerical coefficients that figure in these results are expressed in terms of a single constant S (asymmetry or skewness factor), the value of which has been experimentally determined by Townsend (reference 4). The nondimensional magnitude S (as well as the magnitudes ε and ν) enters only in the expression for the characteristic scales so that with an accuracy up to the choice of units the measurements of the structure of all the fields considered under the assumed hypotheses are described by universal functions not depending on any experimental data (see figs. 1 to 3; the meaning of these functions will be explained in a later discussion).

The investigation of the structure of the velocity field (section 1) is the work of A. M. Obukhoff; the investigation of the pressure field (section 2) was started by Obukhoff (reference 16) and continued by A. M. Yaglom; the investigation of the acceleration field (section 3) was carried out by Yaglom. Several results of the present work were first published in the form of separate short communications (references 7, 16, and 17).

1. Computation of structural functions of velocity field. In order to be able to make use of the concepts of locally isotropic turbulence in investigating the velocity field of a turbulent flow, it is first necessary to separate out those characteristics of the field which depend only on the local structure. The true velocity v will essentially be determined by the mean flow. In the theory of turbulence the usual decomposition of the true velocity v into the mean velocity $\overline{v}$ and the fluctuating velocity $v' = v - \overline{v}$ gives a component v' not depending on this mean flow; but the theory does not solve the problem proposed since the value of v' will be determined mainly by the very

[6]The acceleration of the flow is considered herein to be the total acceleration dv/dt of the fluid particles moving in space.

large vortices, the scale of which is comparable with L. However, as was first noted by A. N. Kolmogoroff (reference 1), the above mentioned required that a separation of the characteristics be effected by considering the difference of the velocities at two sufficiently near points (i.e., the relative motion of two neighboring elements of the fluid). It is clear that this difference will not be affected by the large vortices which transport the pair of points under consideration as a whole. Hence, in the theory of local isotropic turbulence, the following functions are taken as the fundamental quantitative characteristics of the structure of the velocity field:

$$D_{ij}(M,M') = \overline{\left[v_i(M') - v_i(M)\right]\left[v_j(M') - v_j(M)\right]} \quad (i,j = 1,2,3) \quad (1.1)$$

where $v_i(M)$ is the i^{th} component of the velocity vector $v(M)$ at the point M, and the bar above a symbol denotes the average value. The function $D_{ij}(M,M')$ is termed the structural function of the velocity field. According to the preceding discussion, for a distance r between the points M and M' much less than L, this function depends only on the local structure of the flow. On account of the homogeneity and isotropy of the motion of the vortices with scales much less than L, the function $D_{ij}(M,M')$, for $r << L$, is an invariant tensor function of the vector MM' and may therefore be represented in the form

$$D_{ij}(M,M') = A(r)\xi_i\xi_j + B(r)\delta_{ij} \quad (1.2)$$

where ξ_1, ξ_2, and ξ_3 are the components of the vector MM' (so that $\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} = r$) and $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

When first $v_1 = v_j = v_n$ where v_n is the projection of the velocity vector on a certain direction perpendicular to the vector MM' and then $v_i = v_j = v_l$ where v_l is the projection of v on the direction of the vector MM' are set into this formula, it is readily shown that equation (1.2) may be represented in the form

$$D_{ij}(M,M') = \frac{D_{ll}(r) - D_{nn}(r)}{r^2}\xi_i\xi_j + D_{nn}(r)\delta_{ij} \quad (1.3)$$

where the functions $D_{ll}(r)$ and $D_{nn}(r)$ (the longitudinal and transverse structural functions) have the simple physical meaning:

$$D_{ll}(r) = \overline{\left[v_l(M') - v_l(M)\right]^2}$$

$$D_{nn}(r) = \overline{\left[v_n(M') - v_n(M)\right]^2} \tag{1.4}$$

The determination of these functions, $D_{ll}(r)$ and $D_{nn}(r)$, will be the main object of this section.[7]

In the theory of local isotropic turbulence it is possible to consider the functions $D_{ll}(r)$ and $D_{nn}(r)$ as independent of the time. As a matter of fact, a quasi-stationary statistical regime in a region of sufficiently small turbulence scale is assumed. From the considerations of the theory of similarity, it follows that in the range of applicability of the theory of locally isotropic turbulence (i.e., for $r<<L$), the functions $D_{ll}(r)$ and $D_{nn}(r)$ are representable in the form

$$D_{ll}(r) = \sqrt{\nu\varepsilon}\, d_{ll}\left(\frac{r}{\eta}\right)$$

$$D_{nn}(r) = \sqrt{\nu\varepsilon}\, d_{nn}\left(\frac{r}{\eta}\right) \tag{1.5}$$

where $\eta = (\nu^3\varepsilon^{-1})^{1/4}$ is the internal scale of turbulence and $d_{ll}(x)$ and $d_{nn}(x)$ are universal functions. Formulas (1.5) may also be represented in the form

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[7] In the theory of isotropic turbulence, the correlation functions (longitudinal and transverse) are usually employed.

$$B_{ll}(r) = \overline{v_l(M)v_l(M')}$$

$$B_{nn}(r) = \overline{v_n(M)v_n(M')}$$

The structural functions in the isotropic case are connected with the correlation functions by the following relations:

$$D_{ll}(r) = 2(B(0) - B_{ll}(r))$$

$$D_{nn}(r) = 2(B(0) - B_{nn}(r))$$

where $B(0) = B_{ll}(0) = B_{nn}(0)$.

$$D_{ll}(r) = u_1^2 \beta_{ll}\left(\frac{r}{\eta_1}\right)$$
$$D_{nn}(r) = u_1^2 \beta_{nn}\left(\frac{r}{\eta_1}\right) \tag{1.6}$$

where

$$\eta_1 = k_1 \sqrt[4]{\frac{\nu^3}{\varepsilon}}$$
$$u_1 = k_2 \sqrt[4]{\nu\varepsilon} \tag{1.7}$$

The numerical factors k_1 and k_2 can be chosen by inspection and will always be assumed to be of the order of unity, and $\beta_{ll}(x)$ and $\beta_{nn}(x)$ are new universal functions the graphs of which are obtained from the graphs of the functions $d_{ll}(x)$ and $d_{nn}(x)$ by a simple change of scales along the x and y axes.

Since for $r>>\eta$ the functions $D_{ll}(r)$ and $D_{nn}(r)$, on account of the stated physical considerations, should not depend on the viscosity ν, the asymptotic equations should hold

$$d_{ll}(x) \sim x^{2/3}$$
$$\text{for } x>>1 \tag{1.8}$$
$$d_{nn}(x) \sim x^{2/3}$$

The same equations also hold, of course, in relation to the functions $\beta_{ll}(x)$ and $\beta_{nn}(x)$. Whence it follows that for $r>>\eta$

$$D_{ll}(r) \approx C\varepsilon^{2/3} r^{2/3}$$
$$D_{nn}(r) \approx C'\varepsilon^{2/3} r^{2/3} \tag{1.9}$$

(the so-called 2/3 law). In the other extreme case, for $r << \eta$, the difference of the velocities $v(M') - v(M)$ will be of the first order of smallness with respect to r (for such distances the velocity at a point of the flow is continuous and is a differentiable function of the coordinates), so that in this case

$$D_{ll}(r) \approx Ar^2$$
$$D_{nn}(r) \approx A'r^2 \tag{1.10}$$

The more complete theory based on the equations of hydrodynamics is now discussed. First use of the equation of continuity

$$\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = 0 \tag{1.11}$$

shows with little difficulty that

$$D_{nn}(r) = D_{ll}(r) + \frac{r}{2} \frac{dD_{ll}(r)}{dr} \tag{1.12}$$

and that

$$\overline{[v(M') - v(M)][p(M') - p(M)]} = 0 \tag{1.13}$$

where $p(M)$ is the pressure at point M (see, for exmaple, references 2 and 15 and compare also references 18 and 19). Now with the aid of equations (1.12) and (1.13) and the equations of motion

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \Delta v_i \quad (i = 1,2,3) \tag{1.14}$$

it may be shown that the function $D_{lll}(r)$ is connected with the structural function of the third order

$$D_{lll}(r) = \overline{\left[v_l(M') - v_l(M)\right]^3} \tag{1.15}$$

by the known relation of Kolmogoroff (reference 2).[8]

$$D_{lll}(r) - 6\nu \frac{dD_{ll}(r)}{dr} = -\frac{4}{5}\varepsilon r \tag{1.16}$$

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[8]In the case of homogeneous and isotropic turbulence, the equation relative to the correlation functions (references 18 and 19) is easily derived from equation (1.14):

$$\frac{\partial B_{ll}}{\partial t} = \left(\frac{\partial B_{lll}}{\partial r} + \frac{4}{r} B_{lll}\right) + 2\nu\left(\frac{\partial^2 B_{ll}}{\partial r^2} + \frac{4}{r}\frac{\partial B_{ll}}{\partial r}\right)$$

where

$$B_{lll} = \overline{v_l^2(M)v_l(M')}$$

When the correlation functions are replaced by the structural functions given by the formulas in the previous footnote (and by an analogous formula for B_{lll}), the following is obtained:

$$-4\varepsilon = \left(\frac{\partial}{\partial r} + \frac{4}{r}\right)\left(D_{lll} - 6\nu \frac{dD_{ll}}{\partial r}\right) \qquad \left(\varepsilon = -\frac{3}{2}\frac{dB(0)}{dt}\right)$$

from which equation (1.16) is obtained after a single integration with respect to r. In a similar manner, equations (1.12) and (1.13) may be obtained from known results relative to isotropic turbulence. It may likewise be shown that equations (1.12), (1.13), and (1.16) are also valid within the framework of the theory of a locally isotropic flow.

For $r << \eta$, the term $D_{lll}(r)$ may be neglected in this relation (since for these values of r the function $D_{lll}(r)$ will be of third-order smallness with respect to r) and therefore, equations (1.16) and (1.12) give the solutions

$$D_{ll}(r) = \frac{1}{15} \frac{\varepsilon}{\nu} r^2$$

$$\text{for } r << \eta \qquad (1.17)$$

$$D_{nn}(r) = \frac{2}{15} \frac{\varepsilon}{\nu} r^2$$

This is an improvement in the accuracy of relations (1.10). On the other hand, for $r >> \eta$ the term with the viscosity may be rejected since

$$D_{lll}(r) = -\frac{4}{5} \varepsilon r \quad \text{for } r >> \eta \qquad (1.18)$$

The nondimensional magnitude, the asymmetry of distribution of the probabilities for the longitudinal component of the velocity difference is now introduced

$$S = \frac{D_{lll}(r)}{\left[D_{ll}(r)\right]^{3/2}} \qquad (1.19)$$

From the considerations of the theory of dimensions, it follows that for $r >> \eta$ the magnitude S should have a constant value (it can depend only on r and on ε, but from these two magnitudes it is not possible to obtain any nondimensional combination). From equations (1.19), (1.18), and (1.12) it follows that for $r >> \eta$

$$D_{ll}(r) = \left(-\frac{4}{5S}\right)^{2/3} \varepsilon^{2/3} r^{2/3}$$

$$D_{nn}(r) = \frac{4}{3}\left(-\frac{4}{5S}\right)^{2/3} \varepsilon^{2/3} r^{2/3} \tag{1.20}$$

The coefficients C and C' of formulas (1.9) are thus connected with the asymmetry S by the following simple relations:

$$C = \left(-\frac{4}{5S}\right)^{2/3}$$

$$C' = \frac{4}{3} C \tag{1.21}$$

It follows that S is always negative: $S = -|S|$. Formulas (1.17), (1.9), and (1.21) were obtained by A. N. Kolmogoroff in 1941 (references 1 and 2). Up to that time, the results obtained from the equations of hydrodynamics only slightly improved the accuracy of the results obtained previously from a dimensional analysis and they referred only to the two extreme cases: $r >> \eta$ and $r << \eta$. In the matter of the computation of $D_{ll}(r)$ for the intermediate values of r, the single relation (1.16) is of course not sufficient. In this relation are two unknown functions $D_{ll}(r)$ and $D_{lll}(r)$, and therefore still another relation between them is required for their determination. The theory does not give this needed relation, but an attempt may be made to derive it from experimental data.

At the present time, results are known of the direct measurements of the magnitude S for various distances, conducted by Townsend (reference 4) in wind-tunnel tests at very high Reynolds numbers for the purpose of checking the theory of Kolmogoroff. These measurements have shown that the asymmetry S may, with a sufficient degree of accuracy, be assumed as constant not only for $r >> \eta$ but in general for all values of r lying within the range of applicability of the theory of locally isotropic turbulence. The experimental value of S

for all values of r is approximately -0.4.[9] This experimental fact provides the additional relation between $D_{ll}(r)$ and $D_{lll}(r)$, which permits the determination of these functions uniquely for all values of r.

Thus the asymmetry S is assumed constant. From equations (1.16) and (1.19)

$$6\nu \frac{dD_{ll}}{dr} + |S| \left[D_{ll}(r)\right]^{3/2} = \frac{4}{5} \varepsilon r \tag{1.22}$$

where $|S|$ is constant. This equation in the function $D_{ll}(r)$ with coefficients depending on ν, ε, and $|S|$ is considerably simplified if transfer is made to nondimensional magnitudes and the as yet undetermined numerical factors k_1 and k_2 are in the expressions for the scales (i.e., use is made of formulas (1.6) and (1.7)). Then for $\beta_{ll}(x)$,

$$6 \frac{k_2^2}{k_1} \frac{d\beta_{ll}}{dx} + |S| k_2^3 \left[\beta_{ll}(x)\right]^{3/2} = \frac{4}{5} k_1 x \tag{1.23}$$

The magnitudes ε and ν no longer enter into this equation. For a corresponding choice of the constants k_1 and k_2, it is also possible to eliminate the experimental constant $|S|$ and obtain for $\beta_{ll}(x)$ an equation with numerical coefficients. It is convenient to choose k_1 and k_2 such that

$$\begin{aligned} \frac{|S| k_1 k_2}{6} &= \left(\frac{4}{3}\right)^{3/2} \\ \frac{2}{15} \frac{k_1^2}{k_2^2} &= 1 \end{aligned} \tag{1.24}$$

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[9] The experimentally determined values of S fluctuate between the limits -0.36 and -0.42. This scatter lies within the limits of accuracy of the measurements. As the most probable value of S Townsend gives the value -0.38. However, this value may not be assumed reliable for purposes of this report.

that is, to set

$$k_1 = \frac{\sqrt[4]{5}}{\sqrt[4]{2}} \frac{1}{|S|} \approx \frac{5.035}{|S|}$$

$$k_2 = \frac{\sqrt[4]{2}}{\sqrt[4]{45}} \frac{1}{|S|} \approx \frac{1.838}{|S|} \tag{1.25}$$

The equation for $\beta_{ll}(x)$ is then

$$\frac{d\beta_{ll}(x)}{dx} + \left[\frac{4}{3}\beta_{ll}(x)\right]^{3/2} = x \tag{1.26}$$

Equation (1.26) together with the initial condition $\beta_{ll}(0) = 0$ uniquely determines the nondimensional longitudinal structural function $\beta_{ll}(x)$ which describes the structure of the velocity field.[10]

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[10]The structure of a turbulent flow may likewise be described with the aid of the spectral energy distribution. In this case, $E(p)$ denotes the energy of the system of disturbances the wave number of which is larger than p (the scale of disturbance is inversely proportional to the wave number). In the statistical theory of homogeneous (stationary) processes and fields, it is shown that there exists a one to one correspondence between the correlational (structural) functions and the functions $E(p)$; the formulas that permit expressing one of these functions in terms of the other approximate in type the Fourier transformation (cf. references 20 and 31). The 2/3 law for the structural functions, equations (1.9), is equivalent to the ratio of the spectral function $E(p)$ for $p \ll p_1$, to the magnitude $p^{-2/3}$ (i.e., the ratio of the spectral density $dE(p)/dp = E'(p)$ to the magnitude $p^{-5/3}$). The scale η corresponds in the spectral theory to the critical wave number $p_1 = 1/\eta$. The 2/3 law was first obtained in this form by A. M. Obukhoff (reference 3) in 1941. The complete description given in the text of the structural function $D_{ll}(r)$ is equivalent to the determination of the spectral function not only for $p \ll p_1$ but also, in general, for all values of p. There are a number of attempts (references 3, 21, 22 and 5) at a direct theoretical computation of the function $E(p)$ for all p. The results thereby obtained are however difficult to compare with experimental data.

The corresponding nondimensional transverse structural function $\beta_{nn}(x)$ is determined from the relation (1.12) which, after substitution from equation (1.6), may be represented in the form

$$\beta_{nn}(x) = \beta_{ll}(x) + \frac{x}{2} \frac{d\beta_{ll}(x)}{dx} \tag{1.27}$$

Figure 1 shows the graphs of the graphs of the functions $\beta_{ll}(x)$ and $\beta_{nn}(x)$, where $\beta_{ll}(x)$ was determined with the aid of numerical integration[11] of equation (1.26) for the conditions $\beta_{ll}(0) = 0$, and $\beta_{nn}(x)$ was computed with the aid of $\beta_{ll}(x)$ from relation (1.27). The dotted curves denote the asymptotic values of these functions for small and large values of x:

$$\beta_{ll}(x) \approx \frac{1}{2} x^2$$

$$\beta_{nn}(x) \approx x^2 \qquad \text{for } x \ll 1 \tag{1.28}$$

$$\beta_{ll}(x) \approx \frac{3}{4} x^{2/3}$$

$$\beta_{nn}(x) \approx x^{2/3} \qquad \text{for } x \gg 1 \tag{1.29}$$

These formulas correspond to the asymptotic equations (1.17) and (1.20) for the structural functions. The particularly simple form of the asymptotic formulas for the function $\beta_{nn}(x)$ permits a very simple determination of the magnitudes of η_1 and u_1 of equation (1.6) from the transverse structural function $\beta_{nn}(x)$ which was obtained from

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[11] For large values of x (for $x > 8$), it is convenient to make use of the asymptotic expansion for $\beta_{ll}(x)$:

$$\beta_{ll}(x) \sim \frac{3}{4} x^{2/3} \left(1 - \frac{1}{3} x^{-4/3} - \frac{5}{36} x^{-8/3} + \ldots \right)$$

experiment.[12] It is for this reason that the previously mentioned values for the coefficients k_1 and k_2 were chosen.

A direct comparison of the computed curves with the experimental curves obtained in wind-tunnel measurements is technically difficult to make because of the smallness of the scale η. In wind-tunnel measurements it is thus usually possible only to check the agreement with the 2/3 law (see for example references 4 and 5). With relation to the results which refer to the trend of the curve for $r \sim \eta_1$, it is necessary to be satisfied with an indirect check of the type used in checking the accuracy of the constancy of the asymmetry factor. From this point of view measurements in the free atmosphere are evidently more convenient because here the scale η_1 is somewhat larger (of the order of several mm). Nevertheless, such experiments are very complicated and up to this time only one investigation containing data referring to scales of the order of η_1 is known. This is the investigation of Gödecke (reference 23) in which the mean absolute differences in velocity in a direction perpendicular to the base (which corresponds to the transverse structural function) is measured for distances of r varying from 0.1 to 80 centimeters at an altitude of 1.15 meters [?]. The evaluation of these data (reference 7) has shown that they are in good agreement with the theoretical curve obtained herein for $\beta_{nn}(x)$ where $\eta_1 = 0.54$ centimeter and $u_1 = 2.02$ centimeters per second.

2. Computation of structural function of pressure field. The study of the local structure of the field of pressures in a turbulent

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[12] Technically, the measurement of $D_{nn}(r)$ can be affected much more simply than the measurement of $D_{ll}(r)$. For this reason $D_{nn}(r)$ is generally measured in experimental work. Approximation of the curve obtained for $D_{nn}(r)$ to a parabola for small values of r to a parabola and to the 2/3 law for large values of r gives precisely the magnitudes of η_1 and u_1^2, the coordinates of the point of intersection of these two asymptotic expressions. The above construction is conveniently carried out on logarithmic scale; the parabola and the 2/3 law are thereby represented by two straight lines (cf. reference 7).

flow is considered in this section.[13] As a quantitative characteristic of this structure, as in the case of the velocity field, the corresponding structural function is chosen

$$\Pi(M,M') = \overline{\left[p(M') - p(M)\right]^2} \tag{2.1}$$

In the case of a locally isotropic flow, the function $\Pi(M,M')$, for a distance r between the points M and M' much less than the external scale of turbulence L, will depend only on r:

$$\Pi(M,M') = \Pi(r) \tag{2.2}$$

and will be entirely determined by the local structure of the flow.

From considerations of the theory of dimensions it follows that

$$\Pi(r) = q_1^2 \pi\left(\frac{r}{\eta_1}\right) \tag{2.3}$$

where

$$\eta_1 = k_1 \sqrt[4]{\frac{\nu^3}{\varepsilon}}$$

$$q_1 = \rho u_1^2 = k_2^2 \rho \sqrt{\nu\varepsilon} \tag{2.4}$$

the numerical coefficients k_1 and k_2 being assumed to coincide with the coefficients in equation (1.25) and $\pi(x)$ being a universal function. Further, since for $r >> \eta_1$ the structural formula $\Pi(r)$ should not depend on the viscosity ν, the asymptotic equation is

$$\pi(x) \sim x^{4/3} \qquad \text{for} \quad x >> 1 \tag{2.5}$$

and therefore

$$\Pi(r) \sim \rho^2 \varepsilon^{4/3} r^{4/3} \sim \rho^2 \left[D_{ll}(r)\right]^2 \quad \text{for} \quad r >> \eta_1 \tag{2.6}$$

---

[13] From the fact that when deriving the fundamental equation connecting the second and third moments of the velocity field of an isotropic (locally isotropic) incompressible flow, the pressure is excluded (see references 18 and 19 and also equation (1.13)), it does not follow that in an isotropic (locally isotropic) turbulent flow fluctuations of the pressure are absent. Such an erroneous conclusion has been drawn by M. D. Millionshtchikov (reference 24).

It will now be shown how the numerical coefficient in this formula and, in general, the entire trend of the function $\pi(x)$ may be approximately computed.

For this purpose use is made of equations (1.14). If the i^{th} equation is differentiated with respect to x_i and summed over i, then on account of relation (1.11) the terms with $\partial v_i/\partial t$ and with Δv_i drop out and

$$\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(v_j \frac{\partial v_i}{\partial x_j}\right) = -\frac{1}{\rho}\Delta p \tag{2.7}$$

or

$$\Delta p = -\rho \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} \tag{2.8}$$

(the equation of continuity is again applied).

From equation (2.8) it is not difficult to derive the differential equation for the function $\Pi(r)$. It is simplest to proceed as follows. At first the assumption is made that the velocity field and pressure field are statistically homogeneous and isotropic (and not only locally homogeneous and locally isotropic). In this case, the left and right sides of equation (2.8), written out for the point M with coordinates x_1, x_2, x_3, are multiplied correspondingly by the left and right sides of the analogous equation for the point M' with coordinates x_1', x_2', x_3', and the result is averaged and after taking into account the fact that in the case of a homogeneous and isotropic pressure field

$$\overline{\Delta p(M)\Delta p(M')} = \Delta^2\overline{\left[p(M)p(M')\right]}$$

where when differentiation is carried out on the right side with respect to the components $\xi_i = x_i' - x_i$ of the vector MM'

$$\Delta^2\overline{p(M)p(M')} = \rho^2 \sum_{i,j,k,l} \overline{\frac{\partial v_i(M)}{\partial x_j}\frac{\partial v_j(M)}{\partial x_i}\frac{\partial v_k(M')}{\partial x_l'}\frac{\partial v_l(M')}{\partial x_k'}} \tag{2.9}$$

It should now be noted that in the case of a homogeneous and isotropic flow the correlation function $\overline{p(M)p(M')}$ is connected with the structural function (2.1) by the relation (see previous footnote):

$$\Pi(r) = 2\left[\overline{p^2} - \overline{p(M)p(M')}\right] \tag{2.10}$$

Equation (2.9) may therefore be rewritten in the form

$$\Delta^2\Pi(r) \equiv \frac{d^4\Pi(r)}{dr^4} + \frac{4}{r}\frac{d^3\Pi(r)}{dr^3}$$

$$= -2\rho^2 \sum_{i,j,k,l} \overline{\frac{\partial v_i(M)}{\partial x_j}\frac{\partial v_j(M)}{\partial x_i}\frac{\partial v_k(M')}{\partial x'_l}\frac{\partial v_l(M')}{\partial x'_k}} \tag{2.11}$$

This is the required equation. It also has a meaning in the case of locally homogeneous and locally isotropic (but not homogeneous and isotropic) flow, and with the aid of more complicated considerations it may also be derived without the assumption of homogeneity and isotropy.

The structural function $\Pi(r)$ is thus seen to be a solution of equation (2.11), in the right side of which appears a combination of four moments of the derivatives of the velocity field. Unfortunately these moments are not known, and in order that any use may be derived from equation (2.11), it is necessary to make an additional assumption which will permit computing these moments. The assumption adopted herein is that proposed by M. D. Millionshtchikov (reference 24) which states that the fourth moments of the velocity field are expressed in terms of the second moments in the same manner as in the case of the normal Gaussian distribution.[14] As a first approximation this assumption appears to be an entirely natural one. This assumption finds a certain justification in the measurements of Townsend (reference 4) which show that the experimental value of the fourth moment for the velocity derivative $\partial v_1/\partial x_1$ differs by no more than 15 percent from the value computed by the measured value of the second moment on the assumption of normal distribution.

For any four chance magnitudes w_1, w_2, w_3, and w_4 subject to a four-dimensional normal-distribution law, the equation holds (see for example, reference 25):

$$\overline{w_1w_2w_3w_4} = \overline{w_1w_2}\,\overline{w_3w_4} + \overline{w_1w_3}\,\overline{w_2w_4} + \overline{w_1w_4}\,\overline{w_2w_3}$$

[14] It is noted that in the recent work of Heisenberg (reference 21) a hypothesis with regard to the spectral functions of an isotropic turbulent flow precisely equivalent to that proposed by M. D. Millionsctchikov was used.

When this formula is applied to the product of the four derivatives of the velocity field which enter into the right side of equation (2.11), the following equation is obtained:

$$\sum_{i,j,k,l} \overline{\frac{\partial v_i(M)}{\partial x_j} \frac{\partial v_j(M)}{\partial x_i} \frac{\partial v_k(M')}{\partial x'_l} \frac{\partial v_l(M')}{\partial x'_k}} = \sum_{i,j,k,l} \overline{\frac{\partial v_i(M)}{\partial x_j} \frac{\partial v_j(M)}{\partial x_i} \frac{\partial v_k(M')}{\partial x'_l} \frac{\partial v_l(M')}{\partial x'_k}} +$$

$$\sum_{i,j,k,l} \overline{\frac{\partial v_i(M)}{\partial x_j} \frac{\partial v_k(M')}{\partial x'_l}} \overline{\frac{\partial v_j(M)}{\partial x_i} \frac{\partial v_l(M')}{\partial x'_k}} + \sum_{i,j,k,l} \overline{\frac{\partial v_i(M)}{\partial x_j} \frac{\partial v_l(M')}{\partial x'_k}} \overline{\frac{\partial v_j(M)}{\partial x_i} \frac{\partial v_k(M')}{\partial x'_l}} \tag{2.12}$$

The first term on the right-hand side of the equation is proportional to $\overline{\Delta p(M)\, \Delta p(M')}$. In the case of a locally isotropic flow, it is easily verified that this term becomes zero, as can be derived, for example, from equations (1.3) and (1.12). The last two terms of equation (2.12) are equal to each other. It is further noted that in the case of a locally isotropic flow

$$\overline{\frac{\partial v_i(M)}{\partial x_j} \frac{\partial v_k(M')}{\partial x'_l}} = \frac{1}{2} \frac{\partial^2 D_{ik}(M,M')}{\partial \xi_j \partial \xi_l} \tag{2.13}$$

where D_{ik} is the structural function in equation (1.1) and $\xi_j = x'_j - x_j$. From this it follows that for the assumption made about the relation of the second and fourth moments, equation (2.11) may be represented in the form

$$\frac{d^4 \Pi(r)}{dr^4} + \frac{4}{r} \frac{d^3 \Pi(r)}{dr^3} = -\rho^2 \sum_{i,j,k,l} \frac{\partial^2 D_{ik}}{\partial \xi_j \partial \xi_l} \frac{\partial^2 D_{jl}}{\partial \xi_i \partial \xi_k} \tag{2.14}$$

The function on the right side of this equation depends, of course, only on r

$$\Phi(M,M') = \sum_{i,j,k,l} \frac{\partial^2 D_{ik}(M,M')}{\partial \xi_j \partial \xi_l} \frac{\partial^2 D_{jl}(M,M')}{\partial \xi_i \partial \xi_k} = \Phi(r) \tag{2.15}$$

With the aid of equations (1.3) and (1.12), equation (2.15) may be reduced, after rather long transformations, to the form

$$\Phi(r) = \frac{6}{r^2}\left(\frac{dD_{ll}}{dr}\right)^2 + \frac{20}{r}\frac{dD_{ll}}{dr}\frac{d^2D_{ll}}{dr^2} + 4\left(\frac{d^2D_{ll}}{dr^2}\right)^2 + 4\frac{dD_{ll}}{dr}\frac{D^3D_{ll}}{dr^3} \quad (2.16)$$

In accordance with the definition (2.1), the function $\Pi(r)$ is even and assumes the value zero for $r = 0$. Therefore, the two boundary conditions which result are:

$$\Pi(0) = 0 \quad (2.17)$$

$$\Pi'(0) = 0 \quad (2.18)$$

As a third boundary condition use is made of[15]

$$\frac{\Pi(r)}{r^2} \to 0 \qquad \text{for } r \to \infty \quad (2.19)$$

Equation (2.14), for the conditions of equations (2.17), (2.18), and (2.19), has a unique solution which is the required structural function.[16]

Since all linearly independent solutions of the homogeneous equation corresponding to equation (2.14) are found without difficulty (they are 1, r, r^2 and r^{-1}), the required solution of the nonhomogeneous equation can be constructed with the aid of Green's functions. It is easily verified that in the case of the boundary conditions expressed in equations (2.17), (2.18), and (2.19), this function for equation (2.14) has the form

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[15]It may be shown that this condition is required so that the correlation between the differences in the values of the pressures at two pairs of points will approach zero as one pair of points recedes infinitely from the other (the distance between the points for each pair is assumed to be fixed).

[16]It may appear strange that only three boundary conditions are used, whereas equation (2.14) is of the fourth order. The fact is, however, that equation (2.17) is a double condition: Zero is a singular point of equation (2.14) and therefore one boundary condition will be the requirement that the function have regularity at zero.

$$G(r,\xi) = \begin{cases} -\frac{r^2\xi}{6} & \text{for } r \leqslant \xi \\ \frac{\xi^3}{2} - \frac{r\xi^2}{2} - \frac{\xi^4}{6r} & \text{for } r \geqslant \xi \end{cases} \tag{2.20}$$

The required solution for $\Pi(r)$ may be represented in the form

$$\Pi(r) = -\rho^2 \int_0^\infty G(r,\xi)\Phi(\xi)d\xi \tag{2.21}$$

The function $\Phi(r)$, given by equation (2.16), may, on account of relations (1.6) and (1.7), be represented in the form

$$\Phi(r) = \left(\frac{k_2}{k_1}\right)^4 \frac{\varepsilon^2}{\nu^2} \varphi\left(\frac{r}{\eta_1}\right) \tag{2.22}$$

where k_1, k_2, and η_1 are determined from equations (1.25) and (1.7) and $\varphi(x)$ is the universal function:

$$\varphi(x) = \frac{6}{x^2}\left(\frac{d\beta_{ll}}{dx}\right)^2 + \frac{20}{x}\frac{d\beta_{ll}}{dx}\frac{d^2\beta_{ll}}{dx^2} + 4\left(\frac{d^2\beta_{ll}}{\partial x^2}\right)^2 + 4\frac{d\beta_{ll}}{dx}\frac{d^3\beta_{ll}}{dx^3} \tag{2.23}$$

When equations (2.20) and (2.22) are substituted in equation (2.21) and a change of variables is made (cf. equations (2.3) and (2.4))

$$\Pi(r) = k_2^4 \rho^2 \nu\varepsilon\pi\left(\frac{r}{\eta_1}\right) \tag{2.24}$$

where

$$\pi(x) = \int_0^x \left(-\frac{\zeta^3}{2} + \frac{x\zeta^2}{2} + \frac{\zeta^4}{6x}\right)\varphi(\zeta)d\zeta + \int_x^\infty \frac{x^2\zeta}{6}\varphi(\zeta)d\zeta$$

$$= -\frac{1}{2}\int_0^x \zeta^3\varphi(\zeta)d\zeta + \frac{x}{2}\int_0^x \zeta^2\varphi(\zeta)d\zeta + \frac{1}{6x}\int_0^x \zeta^4\varphi(\zeta)d\zeta + \frac{x^2}{6}\int_x^\infty \zeta\varphi(\zeta)d\zeta \tag{2.25}$$

Thus the universal function $\pi(x)$ of equation (2.3) is connected with the function $\beta_{ll}(x)$, which was computed in the preceding section, by use of relations (2.25) and (2.23). Equation (1.26) expresses the derivative $d\beta_{ll}/dx$ in terms of the function $\beta_{ll}(x)$. When this equation is applied several times, the second and third derivatives of these functions can be expressed in terms of $\beta_{ll}(x)$ and therefore also the function $\varphi(x)$. Thus, with knowledge of the function $\beta_{ll}(x)$ from section 1, $\varphi(x)$ can be determined and all the integrals in equation (2.25) can be numerically computed, that is, the function $\pi(x)$, which determines (due to a relation with equation (2.3)) the structure of the pressure field can be computed. The graph of the function $\pi(x)$ thus obtained is given in figure 2.

The dotted curves in figure 2 show the asymptotic behavior of $\pi(x)$ for small and large values of x. Since the motion of the fluid for scales much smaller than η_1 is laminar, for $x<<1$

$$\pi(x) \approx ax^2 \tag{2.26}$$

(See the analogous derivation for the structural functions of the velocity field.) The coefficient a in this formula can easily be obtained from equation (2.25) as follows. From equation (1.28), $\beta_{ll}(x) \approx x^2/2$ for $x<<1$. Hence the function $\varphi(x)$, for such values of x, may be considered as constant: $\varphi(x) \approx 30$. When this value is substituted in equation (2.25), the term proportional to x^2 gives only the last of the integrals in equation (2.25) and

$$a = \frac{1}{6}\int_0^\infty \zeta\varphi(\zeta)d\zeta \approx 0.8 \tag{2.27}$$

(This value has been obtained with the aid of numerical integration.) The curve

$$\pi(x) = 0.83\ x^2$$

is the first of the asymptotic curves drawn in figure 2.

In the second limiting case, for $x>>1$, equation (1.29) shows $\beta_{ll}(x) \approx 3x^{2/3}/4$, and therefore $\varphi(x) \approx 7x^{-8/3}/18$. Equation (2.25) is now represented in the form

$$\pi(x) = \int_0^x \left(-\frac{\zeta^3}{2} + \frac{x\zeta^2}{2} + \frac{\zeta^4}{6x}\right)\frac{7}{18}\,\zeta^{-8/3}\,d\zeta + \int_x^\infty \frac{x^2\zeta}{6}\,\frac{7}{18}\,\zeta^{-8/3}\,d\zeta +$$

$$\int_0^x \left(-\frac{\zeta^3}{2}+\frac{x\zeta^2}{2}+\frac{\zeta^4}{6x}\right)\left(\varphi(\zeta) - \frac{7}{18}\,\zeta^{-8/3}\right)d\zeta + \int_x^\infty \frac{x^2\zeta}{6}\left(\varphi(\zeta) - \frac{7}{18}\,\zeta^{-8/3}\right)d\zeta$$

$$= \frac{9}{16}\,x^{4/3} + \int_0^x \left(-\frac{\zeta^3}{2} + \frac{x\zeta^2}{2} + \frac{\zeta^4}{6x}\right)\left(\varphi(\zeta) - \frac{7}{18}\,\zeta^{-8/3}\right)d\zeta +$$

$$\int_x^\infty \frac{x^2\zeta}{6}\left(\varphi(\zeta) - \frac{7}{18}\,\zeta^{-8/3}\right)d\zeta \qquad (2.28)$$

It is not difficult to see that the values of the integrals on the right side of equation (2.28) for $x \to \infty$ will not increase any faster than the first degree of x, so that the principal term of the asymptotic formula for $\pi(x)$ will be the term $9x^{4/3}/16$. Thus, the numerical coefficient in equation (2.5) is equal to $9/16$ and the symbol of the asymptotic equation (2.5) means only that

$$\frac{\pi(x)}{\frac{9}{16}\,x^{4/3}} \approx \frac{\pi(x)}{\left[\beta_{ll}(x)\right]^2} \approx 1 \qquad \text{for} \quad x >> 1$$

or

$$\frac{\Pi(r)}{\rho^2\left[D_{ll}(r)\right]^2} \approx 1 \qquad \text{for} \quad r >> \eta_l$$

The difference $\pi(x) - 9x^{4/3}/16$, however, increases without limit as x increases.

To obtain the succeeding terms of the asymptotic formula for $\pi(x)$, equation (2.28) is further transformed:

$$\pi(x)=\frac{9}{16}x^{4/3}+\frac{x}{2}\int_0^\infty \zeta^2\left(\varphi(\zeta)-\frac{7}{18}\zeta^{-8/3}\right)d\zeta-\frac{1}{2}\int_0^\infty \zeta^3\left(\varphi(\zeta)-\frac{7}{18}\zeta^{-8/3}\right)d\zeta+$$

$$\frac{1}{6x}\int_0^\infty \zeta^4\left(\varphi(\zeta)-\frac{7}{18}\zeta^{-8/3}\right)d\zeta+$$

$$\int_x^\infty \left(-\frac{x}{2}\zeta^2+\frac{1}{2}\zeta^3-\frac{1}{6x}\zeta^4+\frac{x^2}{6}\zeta\right)\left(\varphi(\zeta)-\frac{7}{18}\zeta^{-8/3}\right)d\zeta \qquad (2.29)$$

Here the integrals over the range from 0 to ∞ converge very rapidly and may be numerically computed while the last integral over the range from x to ∞ may be evaluated for $x>>1$ with the aid of the asymptotic formula given in a previous note. It should be noted that this integral adds only an insignificant increment to the constant term of the asymptotic formula for $\pi(x)$. Finally, with an accuracy up to terms approaching zero as $x\to\infty$,

$$\pi(x)\approx\frac{9}{16}x^{4/3}-0.08x+0.85 \qquad \text{for} \quad x>>1 \qquad (2.30)$$

This is the equation for the asymptotic curve for large values of x plotted in figure 2.

No knowledge of any experimental data on the structure of the pressure field which could be compared with the results obtained herein is known to the authors. It should be remarked that the computations presented previously show that the mean square values of the differences in pressures are found to be so small, as a rule, that their measurement would be associated with very great experimental difficulties. It does not follow from this, however, that the computation of the structural function of the pressure field is practically useless. In the following section it will be shown that the values of the local pressure gradients thereby obtained are very large so that the accelerations produced by the fluctuations of pressure may play an essential role in processes which arise in turbulent flow.

3. Computation of correlation functions of acceleration field. A study of the acceleration field of the fluid particles in a turbulent flow is now undertaken. This field differs from the fields considered in the previous sections in that the very smallest and not the largest vortices[17] are essentially responsible for values of the acceleration at a point, as is the case for the velocity and pressure fluctuations. For this reason, in the case of the field of accelerations of the local flow structure, not only the statistical characteristics of the difference in values of the field at two points (e.g., the structure function) are determined, but also the statistical characteristics of the values of the field. The most important of these characteristics is the correlation function, the mean value of the product of the values of the field at two points (i.e., in the case under consideration, the mean value of the product of the acceleration components).[18] The computation of this correlation function is the main concern in this section.

The value of the correlation function at zero is determined first, that is, the mean square of the acceleration of a fluid particle at a single point. This magnitude is the numerical characteristic of most interest of the acceleration field. From the equations of motion (1.14), the acceleration components of the fluid particle

$$w_i = \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x} \qquad (i = 1,2,3) \quad (3.1)$$

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[17]From considerations of the theory of dimensions it follows that to vortices of the scale of l, where $l >> \eta$, there corresponds the characteristic period $T_l = (l^2/\varepsilon)^{1/3}$ such that the velocity characteristic for these vortices is equal to $v_l = l/T_l = (\varepsilon l)^{1/3}$ and the characteristic acceleration is $w_l = l/T_l^2 = (\varepsilon^2/l)^{1/3}$. Thus it is observed that when the scale of lengths is decreased, characteristic velocity decreases while the characteristic acceleration increases. From this it follows that the very small vortices of scales $l \lesssim \eta$ are mainly responsible for the value of the acceleration at a point of the flow (for such vortices, the dimensional considerations adduced herein do not correspond, of course, to actual conditions, for the motion of these vortices essentially depends on the viscosity)

[18]It is clear that the correlation function is a more significant characteristic of the field than the structural function. Knowledge of the correlation function always allows determination of the structural function also. The converse does not hold true.

are equal to

$$w_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \Delta v_i \tag{3.2}$$

from which is obtained

$$\sum_{i=1}^{3} \overline{w_i^2} = \frac{1}{\rho^2} \sum_{i=1}^{3} \overline{\left(\frac{\partial p}{\partial x_i}\right)^2} - \frac{2\nu}{\rho} \sum_{i=1}^{3} \overline{\frac{\partial p}{\partial x_i} \Delta v_i} + \nu^2 \sum_{i=1}^{3} \overline{(\Delta v_i)^2} \tag{3.3}$$

The first and third term on the right side of this equation may be expressed, without difficulty, in terms of the structural functions of the velocity and pressure fields, equations (1.1) and (2.1):

$$\sum_{i=1}^{3} \overline{\left(\frac{\partial p}{\partial x_i}\right)^2} = \frac{1}{2} \sum_{i=1}^{3} \frac{\partial^2 \Pi(0)}{\partial \xi_i^2} = \frac{3}{2} \frac{d^2 \Pi(0)}{dr^2} \tag{3.4}$$

$$\sum_{i=1}^{3} \overline{(\Delta v_i)^2} = -\frac{1}{2} \Delta^2 \left(\sum_{i=1}^{3} D_{ii}(0) \right) \tag{3.5}$$

The middle term on the right side may be expressed through the interrelated structural functions

$$D_{ip}(M,M') = \overline{\left[v_i(M') - v_i(M)\right]\left[p(M') - p(M)\right]} \quad (i = 1,2,3) \tag{3.6}$$

of the velocity and pressure fields. Since in the case of incompressible local isotropic flow these functions should be equal to zero (see equation (1.13)), the middle term on the right side of equation (3.3) becomes zero, and therefore

$$\sum_{i=1}^{3} \overline{w_i^2} = \frac{3}{2\rho^2} \frac{d^2 \Pi(0)}{dr^2} - \frac{\nu^2}{2} \Delta^2 \left(\sum_{i=1}^{3} D_{ii}(0) \right) \tag{3.7}$$

But on account of equations (2.24), (1.7), (1.25), (2.26), and (2.27)

$$\frac{d^2\Pi(0)}{dr^2} = \frac{k_2^4}{k_1^2}\rho^2\nu^{-1/2}\varepsilon^{3/2}\pi''(0) = \frac{1}{3}\frac{k_2^4}{k_1^2}\left(\int_0^\infty \zeta\varphi(\zeta)d\zeta\right)\rho^2\nu^{-1/2}\varepsilon^{3/2}$$

$$\approx \frac{0.15}{|S|}4.96\rho^2\nu^{-1/2}\varepsilon^{3/2} \approx \frac{0.74}{|S|}\rho^2\nu^{-1/2}\varepsilon^{3/2} \qquad (3.8)$$

Hence[19]

$$\frac{3}{2\rho^2}\frac{d^2\Pi(0)}{dr^2} = \frac{1}{\rho^2}\sum_{i=1}^{3}\overline{\left(\frac{\partial p}{\partial x_i}\right)^2} \approx \frac{1.1}{|S|}\nu^{-1/2}\varepsilon^{3/2} \qquad (3.9)$$

Further use is made of the fact that for any choice of coordinate systems

$$\sum_{i=1}^{3} D_{ii}(r) = D_{ll}(r) + 2D_{nn}(r) \qquad (3.10)$$

and of equation (1.12), the following is obtained:

$$\Delta^2 \sum_{i=1}^{3} D_{ii}(r) \equiv \left(\frac{d^4}{dr^4} + \frac{4}{r}\frac{d^3}{dr^3}\right)\left(3D_{ll}(r) + r\frac{dD_{ll}(r)}{dr}\right)$$

$$= r\frac{d^5D_{ll}(r)}{dr^5} + 11\frac{d^4D_{ll}(r)}{dr^4} + \frac{24}{r}\frac{d^3D_{ll}(r)}{dr^3} \qquad (3.11)$$

With the aid of formulas (1.6) and (1.7), the change from $D_{ll}(r)$ to the nondimensional function $\beta_{ll}(x)$ gives

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[19]The computation of the magnitude $\overline{|\text{grad } p|^2}$ for locally isotropic turbulence is also contained in the work of Heisenberg (reference 21). The method of Heisenberg is based on the employment of the spectral function $E(p)$ and requires considerably more complicated computations. Moreover, in the final formula of Heisenberg, magnitudes enter which cannot be separately measured in tests.

$$\Delta^2 \sum_{i=1}^{3} D_{ii}(r) = \frac{k_2^2}{k_1^4} \nu^{-5/2} \varepsilon^{3/2} \left\{ \frac{r}{\eta_1} \beta_{ll}^{V}\left(\frac{r}{\eta_1}\right) + 11\beta_{ll}^{IV}\left(\frac{r}{\eta_1}\right) + \frac{24}{r\eta_1} \beta_{ll}'''\left(\frac{r}{\eta_1}\right) \right\} \tag{3.12}$$

It is now noted from equation (1.25) that

$$\frac{k_2^2}{k_1^4} = \frac{|S|\sqrt{2}}{120\sqrt{5}} \tag{3.13}$$

and that $\beta_{ll}(x)$ is an even function of x which may be expanded in the neighborhood of zero in a power series in x^2:

$$\beta_{ll}(x) = b_1 x^2 + b_2 x^4 + \cdots \tag{3.14}$$

From equations (3.12), (3.13), and (3.14) the following is obtained:

$$\Delta^2\left(\sum_{i=1}^{3} D_{ii}(0)\right) = \frac{|S|\sqrt{2}}{120\sqrt{5}} \nu^{-5/2} \varepsilon^{3/2} \, 840 \, b_2 = \frac{7\sqrt{2}|S|}{\sqrt{5}} b_2 \nu^{-5/2} \varepsilon^{3/2} \tag{3.15}$$

By use of this method, only the determination of the coefficient b_2 in equation (3.14) remains. From the first of equations (1.28) it follows that $b_1 = 1/2$. When the expansion (3.14) is substituted in equation (1.26) and the coefficients of r^3 are equated (or, what is equivalent, differentiating equation (1.26) with respect to r three times and then setting $r = 0$), the following equation is readily obtained:

$$b_2 = -\frac{1}{3\sqrt{6}} \tag{3.16}$$

The substitution of this value of b_2 in equation (3.15) gives

$$-\frac{\nu^2}{2}\Delta^2\left(\sum_{i=1}^{3} D_{ii}(0)\right) = \nu^2 \sum_{i=1}^{3} \overline{(\Delta v_i)^2} = \frac{7|S|}{6\sqrt{15}} \nu^{-1/2} \varepsilon^{3/2} \approx 0.3|S|\nu^{-1/2} \varepsilon^{3/2} \tag{3.17}$$

Since $|S| \approx 0.4$, it follows from a comparison of equation (3.9) with equation (3.17) that the acceleration of the fluid particles in a turbulent flow is essentially determined by the fluctuating pressure gradients and not by the friction forces. The term with $\Pi''(0)$ in equation (3.7) is more than 20 times as large as the term depending on the viscosity. It shall be seen that this greatly simplifies the computation of the correlation functions of the acceleration field.

When equations (3.9) and (3.17) are substituted in equation (3.7), the following formula is obtained for the computation of the mean square of the acceleration w_0:

$$w_0^2 = \sum_{i=1}^{3} \overline{(w_i)^2} \approx \left(\frac{1.1}{|S|} + 0.3|S| \right) \nu^{-1/2} \varepsilon^{3/2} \tag{3.18}$$

Since $|S| \approx 0.4$, equation (3.18) may be replaced by the simple relation

$$w_0^2 \approx 3\nu^{-1/2} \varepsilon^{3/2} \tag{3.19}$$

This general relation permits the estimation of the order of magnitude of w_0 in specific cases of turbulent flow without difficulty.

As an example, formula (3.19) is applied to the computation of the mean square acceleration in certain turbulent flows behind a screen (or grid) in wind tunnels and in turbulent atmosphere. In the case where isotropic turbulence was produced by screens in wind tunnels, the dissipation ε may be defined either as

$$\varepsilon = -\frac{3}{2} V \overline{\frac{dv'^2}{dx}}$$

where $\overline{v'^2}$ is the mean square of the velocity fluctuation, V the mean velocity, x the distance from the screen, or as

$$\varepsilon = \frac{15\nu \overline{v'^2}}{\lambda^2}$$

where λ is the length introduced by Taylor, experimentally determinable by inscribing a parabola in the graph of the correlation function $B_{ll}(r)$. When the dissipation ε is known, w_0 can be computed from the formula

$$w_0 = 2.77\ \varepsilon^{3/4}\ \text{cm/sec}^2 \tag{3.20}$$

obtained by substituting the air viscosity $\nu = 0.15$ sq cm/sec in equation (3.19).

In particular, when use is made of some of the data given by Townsend (reference 4) (these data refer to the flow in a wind tunnel behind a square screen with size of mesh $M = 6$ inches at a distance $x = 30.5\ M$ from the screen for various values of the velocity V), the following values for ε and w_0 are obtained:

V m sec^{-1}	ε cm^2 sec^{-3}	w_0 cm sec^{-2}
12.2	60.5	60.4
24.4	312.4	206.8
30.5	559.8	320.3

From this table it is observed that the instantaneous values of the acceleration in turbulent flow behind the screen will be of the order of several meters per second per second.

The application of formula (3.19) or (3.20) to the computation of the accelerations in a turbulent atmosphere is rendered difficult by the fact that at the present time there are no available measurements of energy dissipation for this case. However, for the degree of accuracy of the computations, much justification exists for employing an estimate of the magnitude of ε for a turbulent atmosphere by the formulas of the theory of the logarithmic boundary layer. It is known (reference 15) that for the logarithmic boundary layer

$$\varepsilon = \frac{1}{\varkappa} \frac{v_*^3}{y} \tag{3.21}$$

where y is the distance from the wall, $\varkappa$ is a nondimensional constant (Kármán constant) equal approximately to 0.4, and $v_* = \sqrt{\tau_0/\rho}$ (τ_0 is the friction stress, ρ the density) is the so-called dynamic velocity determined by the difference of the mean velocities at two points or by the mean velocity at one point and the magnitude of the roughness. Substitution in formula (3.19) of expression (3.21) for the dissipation and $\nu = 0.15$ sq cm/sec gives a computational formula which determines the mean square acceleration in a logarithmic boundary air layer:

$$w_0 \approx 5.5 \frac{v_*^{9/4}}{y^{3/4}} \frac{\text{cm}}{\text{sec}^2} \tag{3.22}$$

Since v_* is proportional to the mean velocity V,

$$w_0 \sim V^{9/4} \tag{3.23}$$

that is, w_0 increases rapidly with V. For the example, the magnitude of the roughness is assumed to be $h_0 = 3$ cm (it is noted incidentally that the computations following depend relatively little on the magnitude of the roughness) and the mean velocity of the wind at the height 150 cm is denoted by V. Then

$$v = \frac{\varkappa V}{\ln(y/h_0)} \approx 0.1\ V \tag{3.24}$$

and for the mean square acceleration w_0 at various velocities V the following values are obtained:

V, m sec^{-1}	1	3	5	6	8
w_0, cm sec^{-2}	22	260	830	1200	2400

The mean square acceleration under the conditions considered for a mean velocity of the wind $V \approx 5.5$ m/sec thus attains the magnitude of the acceleration of gravity g, and for a greater wind velocity may considerably exceed this acceleration. It is natural to assume that such large accelerations may play a significant part in many physical processes in the atmosphere (e.g., in the phenomenon of the condensation of fogs).

The computation of the correlation function of the acceleration field is now considered:

$$A_{ij}(M,M') = \overline{w_i(M)w_j(M')} \tag{3.25}$$

Again, substitution of equation (3.2) gives

$$A_{ij}(M,M') = \frac{1}{\rho^2}\overline{\frac{\partial p}{\partial x_i}\frac{\partial p'}{\partial x'_j}} - \frac{\nu}{\rho}\left(\overline{\frac{\partial p}{\partial x_i}\Delta' v'_j} + \overline{\Delta v_i \frac{\partial p'}{\partial x'_j}}\right) + \nu^2\overline{\Delta v_i \Delta' v'_j} \tag{3.26}$$

The magnitudes without the primes refer to point M and those with primes to the point M'. The middle term on the right side may be neglected for the same reasons for which the middle term on the right side of equation (3.3) was previously rejected, and the first and third

terms may easily be expressed in terms of the structural functions (1.1) and (2.1). Therefore,

$$\overline{\frac{\partial p}{\partial x_i}\frac{\partial p'}{\partial x'_j}} = \frac{1}{2}\frac{\partial^2 \Pi(M,M')}{\partial \xi_i \partial \xi_j} \tag{3.27}$$

where ξ_i and ξ_j are the components of the vector MM' and

$$\overline{\Delta v_i \Delta' v'_j} = -\frac{1}{2}\Delta^2 D_{ij}(M,M') \qquad \left(\Delta = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \frac{\partial^2}{\partial \xi_3^2}\right) \tag{3.28}$$

The transformation of equations (3.27) and (3.28) follows. Since $\Pi(M,M')$ depends only on the distance $r = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$,

$$\frac{\partial^2 \Pi(M,M')}{\partial \xi_i \partial \xi_j} = \frac{\partial}{\partial \xi_i}\left\{\frac{d\Pi(r)}{dr}\frac{\xi_j}{r}\right\} = \left\{\frac{d^2\Pi(r)}{dr^2} - \frac{1}{r}\frac{d\Pi(r)}{dr}\right\}\frac{\xi_i \xi_j}{r^2} + \frac{1}{r}\frac{d\Pi(r)}{dr}\delta_{ij} \tag{3.29}$$

Replacement of $D_{ij}(M,M')$ by means of equations (1.3) and (1.12) yields

$$\left[D_{ll}(r) + \frac{r}{2}D'_{ll}(r)\right]\delta_{ij} - \frac{1}{2r}D'_{ll}(r)\xi_i \xi_j$$

which gives the following:

$$\Delta^2 D_{ij}(M,M') = D_1(r)\frac{\xi_i \xi_j}{r^2} + D_2(r)\delta_{ij} \tag{3.30}$$

where

$$D_1(r) = -\frac{12}{r^3}\frac{dD_{ll}}{dr} + \frac{12}{r^2}\frac{d^2 D_{ll}}{dr^2} - 4\frac{d^4 D_{ll}}{dr^4} - \frac{r}{2}\frac{d^5 D_{ll}}{dr^5} \tag{3.31}$$

$$D_2(r) = \frac{4}{r^3}\frac{dD_{ll}}{dr} - \frac{4}{r^2}\frac{d^2 {}_{ll}}{dr^2} + \frac{8}{r}\frac{d^3 D_{ll}}{dr^3} + 5\frac{d^4 D_{ll}}{dr^4} + \frac{r}{2}\frac{d^5 D_{ll}}{dr^5} \tag{3.32}$$

Thus

$$A_{ij}(M,M') = A_1(r)\frac{\xi_i \xi_j}{r^2} + A_2(r)\delta_{ij} \tag{3.33}$$

where

$$A_1(r) = \frac{1}{2\rho^2}\left\{\frac{d^2\Pi(r)}{dr^2} - \frac{1}{r}\frac{d\Pi(r)}{dr}\right\} - \frac{\nu^2}{2}D_1(r) \tag{3.34}$$

$$A_2(r) = \frac{1}{2\rho^2 r}\frac{d\Pi(r)}{dr} - \frac{\nu^2}{2}D_2(r) \tag{3.35}$$

and $D_1(r)$ and $D_2(r)$ are determined by formulas (3.31) and (3.32).

The functions $A_1(r)$ and $A_2(r)$ are expressed in terms of the longitudinal and transverse correlation functions of the acceleration field determined by the equations

$$\begin{aligned} A_{ll}(r) &= \overline{w_l(M)w_l(M')} \\ A_{nn}(r) &= \overline{w_n(M)w_n(M')} \end{aligned} \tag{3.36}$$

where $w_l(M)$ and $w_l(M')$ are the projections of the accelerations at the points M and M' on the direction of the vector MM', and $w_n(M)$ and $w_n(M')$ are the projections of the accelerations at these points in a direction perpendicular to the vector MM'. In fact, the acceleration field of a locally isotropic turbulent flow is isotropic in the usual sense, and therefore

$$A_{ij}(M,M') = \frac{A_{ll}(r) - A_{nn}(r)}{r^2}\xi_i\xi_j + A_{nn}(r)\delta_{ij} \tag{3.37}$$

(see reference 19 and equation (1.3) herein). Comparing equations (3.33) and (3.37) and taking into account equations (3.34) and (3.35) yields

$$A_{ll}(r) = A_1(r) + A_2(r) = \frac{1}{2\rho^2}\frac{d^2\Pi(r)}{dr^2} - \frac{\nu^2}{2}(D_1(r) + D_2(r)) \tag{3.38}$$

$$A_{nn}(r) = A_2(r) = \frac{1}{2\rho^2 r}\frac{d\Pi(r)}{dr} - \frac{\nu^2}{2}D_2(r) \tag{3.39}$$

In formulas (3.38) and (3.39) it is possible, in the usual manner, to pass to nondimensional functions. These may be further computed with the aid of the results of sections 1 and 2.

It may be noted that in these computations the terms with $D_1(r)$ and $D_2(r)$ may be neglected without introducing any appreciable error. In fact, it was shown previously that for $r = 0$ the terms depending on the viscosity, that is, the terms containing $D_1(r)$ and $D_2(r)$, are negligibly small compared with the terms determining the pressure gradients. With increasing r both terms decrease asymptotically, the terms depending on the viscosity decreasing much more rapidly than those determined by the pressure gradient. From formulas (2.6) and (1.9) it follows that for $r >> \eta_1$

$$\frac{d^2\Pi(r)}{dr^2} \sim r^{-2/3}$$

$$\frac{1}{r}\frac{d\Pi(r)}{dr} \sim r^{-2/3} \tag{3.40}$$

$$D_1(r) \sim r^{-10/3}$$

$$D_2(r) \sim r^{-10/3} \tag{3.41}$$

Thus, for both small and large r, the terms of equations (3.38) and (3.39) containing ν are considerably smaller than the terms depending on $\Pi(r)$. In this connection, the investigation of the structure of the acceleration field in a turbulent flow permits the rejection of terms with viscosity in the equations of motion, and the assumption that

$$w_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} \qquad (i = 1,2,3) \tag{3.42}$$

$$A_{ij}(M,M') = \frac{1}{2\rho^2}\frac{\partial^2\Pi(M,M')}{\partial\xi_i\partial\xi_j} \tag{3.43}$$

For the longitudinal and transverse correlation functions (3.36), there is then obtained

$$A_{ll}(r) = \frac{1}{2\rho^2}\frac{d^2\Pi(r)}{dr^2}$$

$$A_{nn}(r) = \frac{1}{2\rho^2 r}\frac{d\Pi(r)}{dr} \tag{3.44}$$

With the aid of formulas (2.24), (1.7), and (1.25), the change to nondimensional magnitudes is made, and using equation (2.25)

$$A_{ll}(r) = \frac{k_2^4}{k_1^2}\,\nu^{-1/2}\,\varepsilon^{3/2}\alpha_{ll}\left(\frac{r}{\eta_1}\right) \approx \frac{0.45}{|S|}\,\nu^{-1/2}\,\varepsilon^{3/2}\,\alpha_{ll}\left(\frac{r}{\eta_1}\right) \tag{3.45}$$

$$A_{nn}(r) = \frac{k_2^4}{k_1^2}\,\nu^{-1/2}\,\varepsilon^{3/2}\,\alpha_{nn}\left(\frac{r}{\eta_1}\right) \approx \frac{0.45}{|S|}\,\nu^{-1/2}\,\varepsilon^{3/2}\,\alpha_{nn}\left(\frac{r}{\eta_1}\right) \tag{3.46}$$

where $\alpha_{ll}(x)$ and $\alpha_{nn}(x)$ are universal functions which are given by the formulas

$$\alpha_{ll}(x) = \frac{1}{6x^3}\int_0^x \zeta^4\varphi(\zeta)d\zeta + \frac{1}{6}\int_x^\infty \zeta\varphi(\zeta)d\zeta \tag{3.47}$$

$$\alpha_{nn}(x) = \frac{1}{4x}\int_0^x \zeta^2\varphi(\zeta)d\zeta - \frac{1}{12x^3}\int_0^x \zeta^4\varphi(\zeta)d\zeta + \frac{1}{6}\int_x^\infty \zeta\varphi(\zeta)d\zeta \tag{3.48}$$

As in the case of the velocity and pressure fields, for $x<<1$ and for $x>>1$, it is possible to obtain for the functions introduced in the theory described herein simple asymptotic formulas. It is clear first of all that

$$\alpha_{ll}(0) = \alpha_{nn}(0) = \frac{1}{6}\int_0^\infty \zeta\varphi(\zeta)d\zeta \approx 0.83 \tag{3.49}$$

If in formulas (3.47) and (3.48) x is assumed much less than 1 ($x<<1$), use may be made of the fact that for these values of x, as shown in the first formula of equation (1.28), $\beta_{ll}(x) \approx x^2/2$, and therefore $\varphi(x) \approx 30$; whence

$$\alpha_{ll}(x) \approx \alpha_{ll}(0) - \frac{3}{2}x^2$$

$$\alpha_{nn}(x) \approx \alpha_{nn}(0) - \frac{1}{2}x^2 \qquad \text{for } x<<1 \tag{3.50}$$

In the second extreme case, for $x>>1$, the asymptotic behavior of $\alpha_{ll}(x)$ and $\alpha_{nn}(x)$ is determined with the aid of formulas (2.30) and (3.44).

$$\alpha_{ll}(x) \approx \frac{1}{8} x^{-2/3}$$

$$\alpha_{nn}(x) \approx \frac{3}{8} x^{-2/3} \qquad \text{for } x>>1 \qquad (3.51)$$

The computation of the functions $\alpha_{ll}(x)$ and $\alpha_{nn}(x)$ for $x \sim 1$ may be carried out numerically by using the data contained in sections 1 and 2. It is convenient in place of $\alpha_{ll}(x)$ and $\alpha_{nn}(x)$ to introduce the normalized functions

$$R_{ll}(x) = \frac{\alpha_{ll}(x)}{\alpha_{ll}(0)}$$

$$R_{nn}(x) = \frac{\alpha_{nn}(x)}{\alpha_{nn}(0)} \qquad (3.52)$$

These functions are equal respectively to the correlation coefficient of the longitudinal and transverse components of the acceleration at two points a distance $r = x\eta_1$ from each other. The graphs of the functions $R_{ll}(x)$ and $R_{nn}(x)$, which were determined by numerical integration of the integrals appearing in the right sides of equations (3.47) and (3.48), are shown in figure 3. It is seen that the longitudinal correlation function $R_{ll}(x)$ rapidly decreases, and for $x \geq 1.1$ it may practically be considered equal to zero. The function $R_{nn}(x)$, on the contrary, decreases at a relatively slow rate, and for $x = 3$ is approximately equal to 0.17. When the magnitudes of these functions are estimated for relatively large values of x (of the order of 10 and above), formulas (3.51) may be used. From these formulas, when $x = 10$, for example, $R_{ll}(10) \approx 0.03$. (In fig. 3 the range of applicability of formulas (3.51) is not represented, since to do so it would be necessary to choose a much smaller scale.)

It may be noted further that the form of the correlation functions of the acceleration field shown in figure 3 differs sharply from the form of the correlation functions of the velocity field for isotropic turbulence. In the case of the velocity field, the graph of the longitudinal correlation function is generally located above the graph of the transverse function and the axis of the abscissas intersects the second and not the first of these curves. This difference in behavior of the correlation functions for the velocities and accelerations is

explained by the fact that the velocity field in an incompressible fluid is a solenoidal vector field, whereas the acceleration field is considered as a potential vector field (see equation (3.42)). From this it follows that the functions $R_{ll}(x)$ and $R_{nn}(x)$ are interconnected by the relation

$$R_{ll}(x) = R_{nn}(x) + x \frac{dR_{nn}(x)}{dx} \tag{3.53}$$

[This relation, which is a necessary and sufficient condition for the isotropic potential vector field having the correlation functions $R_{ll}(x)$ and $R_{nn}(x)$, was obtained by A. M. Obukhoff, while the correlation functions $B_{ll}(r)$ and $B_{nn}(r)$ of the velocity field satisfy the Karman condition (cf. reference 19 and equation (1.13))]:

$$B_{nn}(r) = B_{ll}(r) + \frac{r}{2} \frac{dB_{ll}(r)}{dr} \tag{3.54}$$

Conditions (3.53) and (3.54), in addition to the factor 1/2 in the second term on the right, differ in the interchange of the roles of the longitudinal and transverse functions. It is not surprising, therefore, that the functions $R_{ll}(x)$ and $R_{nn}(x)$ behave in a manner opposite to the behavior of the functions $B_{ll}(r)$ and $B_{nn}(r)$.

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Translated by S. Reiss
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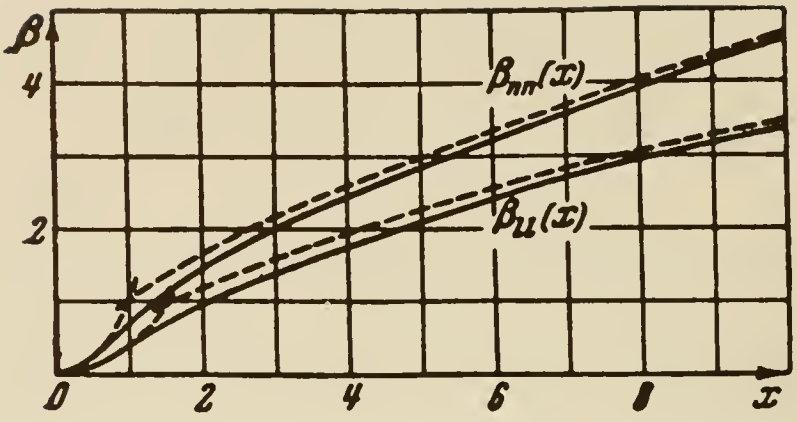

Figure 1

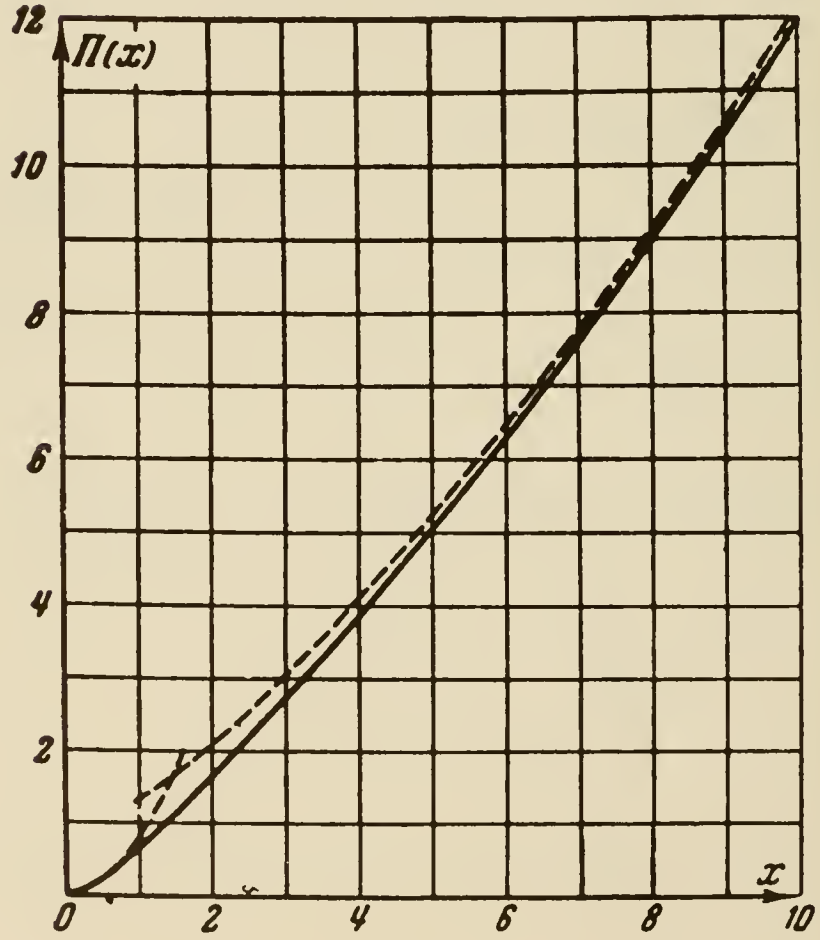

Figure 2

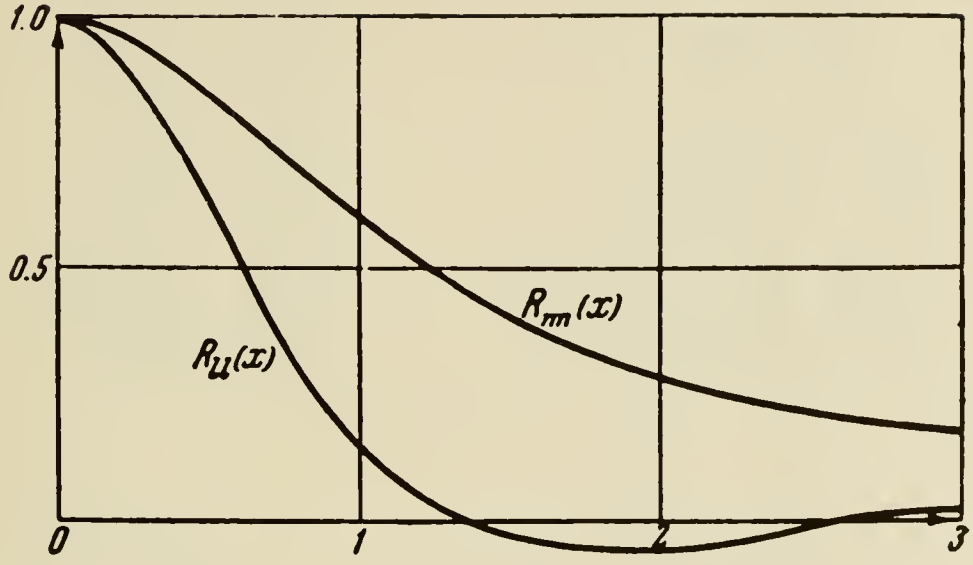

Figure 3

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THE MICROSTRUCTURE OF TURBULENT FLOW.
(Mikrostructura turbulentnogo potoka.) A. M. Obukhoff and A. M. Yaglom. June 1953. 41p. diagrs. (NACA TM 1350. Trans. from Prikladnaya Matematika i Mekhanika, v. 15, 1951, p. 3-26).

An attempt is made to describe quantitatively the structure of the velocity, pressure, and acceleration fields for all scales for which the theory of Kolmogoroff is applicable.

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